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Junction conditions in general relativity

C K Raju†

Physical and Earth Sciences Division, Indian Statistical Institute, Calcutta 700 035, India

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Abstract. Earlier work on the problem of junction conditions is briefly reviewed. An analytical formalism is developed to deal with the occurrence of jump discontinuities in the $g_{\mu\nu}$ or their first derivatives across a hypersurface Σ. It is shown that the equations of relativity remain meaningful *at* Σ, even when Σ does not inherit a unique intrinsic geometry, so that the $g_{\mu\nu}$ are discontinuous across Σ in natural coordinates. The spherically symmetric surface layer at the Schwarzschild–Minkowski junction is used to illustrate these techniques, and to establish rigorously the existence of C^0 solutions of the Einstein equations and the conservation equations. The possible validity of relativity at the microscopic level is examined, and it is concluded that, if relativity is valid at the microscopic level, then it is likely that the $g_{\mu\nu}$ are not globally continuously differentiable.

1. Introduction

The problem of junction conditions in general relativity may be stated thus: at a sharp boundary between matter and empty space–time, how smooth should the 10 distinct functions (of the coordinates) $g_{\mu\nu}$ be? This problem, and its generalisations, occur in the study of thin shells of matter (ejected by a nova, for example), collapsing stars (to model the boundary of the star), gravitational shock waves, and gravitational screening. Fundamentally, a resolution of this problem necessitates an elucidation of the mathematical assumptions underlying the theory of relativity.

The study of this problem was initiated by Lanczos (1922, 1924), who noticed that the problem, as stated above, is not well-posed. It is necessary to distinguish between 'genuine' discontinuities in the derivatives of the metric tensor, and 'spurious' discontinuities arising from a particular choice of the coordinate system. Later work on allied problems highlighted this difficulty by ignoring it and reaching peculiar conclusions (e.g. Raychaudhuri 1953, Israel 1958).

The introduction of admissible coordinates by Lichnerowicz (1955) considerably clarified the situation. The need to introduce admissible coordinates may be understood as follows. A statement of the Einstein equations, and the conservation equations, assumes that the $g_{\mu\nu}$ may be differentiated thrice. That is, it must be posited that space–time admits a C^2 atlas in which the $g_{\mu\nu}$ are thrice continuously differentiable‡. The last condition may be relaxed somewhat since the conservation equations still make sense, in the theory of generalised functions (Gel'fand and Shilov 1964), if the second derivatives of the $g_{\mu\nu}$ have, at most, a *simple* discontinuity across a *smooth* hypersurface. The extension to a finite number of (non-intersecting) hypersurfaces is trivial. This is precisely Lichnerowicz's postulate: that space–time admits a

† Present address: Department of Statistics, University of Poona, Ganeshkhind, Pune 411 007, India.

‡ If a manifold admits a C^2 atlas, it admits a C^∞ or C^ω atlas.

C^2 atlas in which the $g_{\mu\nu}$ are twice continuously differentiable except at certain smooth hypersurfaces where the second derivatives may have a simple discontinuity. The Einstein equations are, then, covariant under admissible coordinate transformations (C^2 diffeomorphisms).

In practice, admissible coordinates may not be the most convenient, and O'Brien and Synge (1952) utilised a scalar invariant to state the junction conditions in the form (Synge 1960)

$$G^{\mu}_{\nu} f_{,\mu} \varphi^{\nu} = [C] \tag{1.1}$$

where G^{μ}_{ν} denotes the Einstein tensor, $f(x)=0$ is the equation of the hypersurface Σ, φ^{ν} is a contravariant vector field that undergoes parallel transport along any pre-assigned set of curves which cross Σ, and $[C]$ indicates any quantity that is continuous across Σ. The invariant may be evaluated in (non-admissible) coordinate systems that do not match continuously on Σ. The coordinate systems, of course, must be related to admissible coordinates by admissible transformations on either side of Σ.

Israel (1966) gave an elegant generalisation of these conditions for the case where the hypersurface Σ is not null. A smooth hypersurface divides space–time into two half-spaces V^+ and V^-, and Israel (1966) defined a singular hypersurface as one which has different extrinsic curvatures (second fundamental forms) associated with its embeddings in V^+ and V^-. If we use gaussian coordinates based on Σ (these are admissible if we accept Lichnerowicz's postulate), then the normal derivatives of the g_{ij} have a jump across Σ, the $g_{\mu\nu}$ themselves being continuous across Σ. The last assertion holds because V^+ and V^- must induce the same intrinsic geometry on Σ (Misner *et al* 1973). That is, in natural coordinates (the analogue of Lichnerowicz's admissible coordinates), the $g_{\mu\nu}$ need only be of class C^0 across a singular hypersurface.

Using the Einstein equations applied to the half-spaces V^+ and V^-, Israel (1966) deduced a three-dimensional law of conservation for surface layers *in vacuo*. As in the case of the O'Brien–Synge junction conditions, one may utilise coordinates which do not match continuously on Σ. Equations, in intrinsic form, for gravitational shocks have been developed by Choquet-Bruhat (1968).

Dautcourt (1964), Papapetrou and Treder (1959) and Papapetrou and Hamoui (1968) have developed an analytic approach to the study of singular hypersurfaces. This formalism applies without modification to null hypersurfaces, or to the case where a surface layer occurs in conjunction with a boundary surface. Papapetrou and Hamoui (1968) obtained the equations of motion for surface layers in four-dimensional form, and proved that the surface energy three-tensor, introduced heuristically by Israel (1966), is just the projection onto Σ of the generalised material energy four-tensor.

The motivation behind the present approach is, firstly, to clarify the status of the equations of relativity *at* a hypersurface of discontinuity. Using Raju's (1982) non-linear theory of distributions, it is shown that the Einstein equations remain meaningful at Σ, even when Σ does not inherit a unique intrinsic geometry ($g_{\mu\nu}$ discontinuous in natural coordinates). Physically, the existence of a hypersurface with a non-unique intrinsic geometry would correspond to the phenomenon of gravitational screening. It is also possible to generalise to the case where the hypersurface is not smooth. Before investigating these generalisations, however, it is necessary to establish the existence of solutions to these equations. This would also serve to clarify the relationship between this approach and earlier approaches. Therefore, in the present paper we establish the existence of C^0 solutions of the Einstein equations, and show that our approach leads to a considerable simplification of the formalism of Papapetrou and Hamoui (1968).

Secondly, we utilise a microphysical criterion (the existence of matter in the form of particles) to investigate the physical validity of postulates regarding admissible coordinates and the smoothness of the $g_{\mu\nu}$. The extrapolation of relativity to the microphysical domain may appear highly questionable, in view of the current opinion that relativity is applicable at the micro-level only after it has been integrated with quantum mechanics. This paper justifies the extrapolation on the grounds of internal consistency and Raju's (1981) interpretation of quantum mechanics as a theory of extended shell-like particles.

2. Algebra of distributions

We begin with the problem regarding the meaning and validity of the equations of relativity when the $g_{\mu\nu}$ or their first derivatives $g_{\mu\nu,\sigma}$ (in a given C^2 atlas) have simple discontinuities across a hypersurface Σ. If only the $g_{\mu\nu,\sigma}$ are discontinuous across Σ, the Christoffel symbols are not well-defined on Σ. If the $g_{\mu\nu}$ are discontinuous across Σ, the Ricci tensor, for instance, would involve entities such as δ^2, δ being the Dirac delta-function. Earlier authors (Dautcourt 1964, Israel 1966, Papapetrou and Hamoui 1968, Choquet-Bruhat 1968) have tackled this problem by replacing the usual equations by a more meaningful set of equations at Σ.

In the present approach we will, instead, assign some meaning to the above entities. The physical aspect of the problem introduces the additional constraint of defining the above entities in such a manner that our belief in the equations of relativity, at the hypersurface of discontinuity, continues to be justified. This problem has been solved by Raju (1982) using the techniques of 'non-standard analysis' to ensure that our continued belief in the Einstein equations would not be reduced to a phenomenological one.

The definitions given in Raju (1982), of products and compositions with distributions, are as follows. We let D, D' denote the space of test functions and distributions, respectively, and $^*D$ and $^*D'$ their corresponding non-standard extensions (Stroyan and Luxemburg 1976). For $f, g \in D'$, define

$$
\begin{aligned}
&f \cdot g = \lim_{n=\omega} {}^*(f_n \cdot g) \\
&f_n = f \otimes \delta_n \qquad\qquad (2.1) \\
&\langle f_n \cdot g, h\rangle = \langle g, f_n \cdot h\rangle \qquad \forall h \in D
\end{aligned}
$$

where $\otimes$ denotes convolution, δ_n is a sequence converging to the δ-function, and $\langle g, h\rangle$ denotes the value of the functional g at h. The * in (2.1) denotes the non-standard extension of the sequence of distributions $f_n \cdot g$, and the notation $\lim_{n=\omega}$ refers to an evaluation of the ωth term of this sequence for a fixed positive infinite integer ω. With the above definition, $f \cdot g$ always exists in $^*D'$. Thus, $\delta^2 = \delta_\omega(0)\delta$ is an infinite distribution.

The final results are independent of the choice of δ_n or ω because we have a theorem that asserts, for instance, that the meaning of the equation

$$a\delta^2 + b\delta + c = 0 \qquad (2.2)$$

for standard real numbers a, b, c is simply that

$$a = b = c = 0. \tag{2.3}$$

We observe that no new phenomenology has been introduced, and that the results derived by using non-standard techniques could very well have been derived without using them (Stroyan and Luxembourg 1976).

The associative and commutative laws fail, in general, although the product may be symmetrised by defining

$$f \odot g = \tfrac{1}{2}(f \cdot g + g \cdot f). \tag{2.4}$$

For multiplication by discontinuous functions, we have the following useful result: if f is a function with a simple discontinuity at 0, then

$$f \cdot \delta = \tfrac{1}{2}(f(0^+) + f(0^-))\delta. \tag{2.5}$$

If g is an infinitely differentiable function and $f \in D'$ compositions are defined by

$$f(g(x)) = \lim_{n=\omega} f_n(g(x)). \tag{2.6}$$

$f(g(x))$ defined by (2.6) exists, and agrees with the usual definition if g has the real roots $x_1, x_2, \ldots, x_n$, with $g'(x_i) \neq 0$ for $1 \leqslant i \leqslant n$. The chain rule is valid, and the definitions remain meaningful even if g has multiple roots.

3. The general formalism

We begin with some notation. If Σ is a smooth hypersurface, let

$$\begin{aligned} \chi^+ &= \chi_{V^+} = \text{characteristic function of } V^+ \\ \chi^- &= 1 - \chi^+. \end{aligned} \tag{3.1}$$

If we use coordinates such that the equation of Σ is $f(x) = 0$, and that of V^+ is $f(x) > 0$, then χ^+ may be written as $\theta(f(x))$ where θ is the Heaviside function.

Any function of the coordinates, $h(x)$, can be written as

$$h = h^+\chi^+ + h^-\chi^-. \tag{3.2}$$

If h has, at most, a simple discontinuity at Σ, we define

$$\begin{aligned} [h](P) &= \lim_{Q \to P} h^+ - \lim_{R \to P} h^- \\ h\,|\,(P) &= \tfrac{1}{2}(\lim_{Q \to P} h^+ + \lim_{R \to P} h^-) \end{aligned} \tag{3.3}$$

where $P \in \Sigma$, and Q and R tend to P through V^+ and V^- respectively. (3.3) makes sense even if the components of the metric tensor have a simple discontinuity at Σ.

The following properties follow immediately from the results stated in § 2.

$$\chi^+ \cdot \chi^+ = \chi^+ \qquad \chi^- \cdot \chi^- = \chi^- \qquad \chi^+ \cdot \chi^- = 0 \qquad \chi^+ \cdot \delta_\Sigma = \tfrac{1}{2}\delta_\Sigma \tag{3.4}$$

where

$$(\chi^+)' = \delta_\Sigma \tag{3.5}$$

i.e. if the equation of Σ is $f(x)=0$, then

$$\chi^{+}_{,\mu}=f_{,\mu}\delta(f(x)). \tag{3.6}$$

For later use we also record

$$\begin{aligned}&[h_1+h_2]=[h_1]+[h_2]\\&[h_1h_2]=h_1|[h_2] \qquad \text{if } [h_1]=0\\&(h_1+h_2)|=h_1|+h_2|\\&(h_1h_2)|=h_1|h_2| \qquad \text{if } [h_1]=0 \quad \text{or } [h_2]=0.\end{aligned} \tag{3.7}$$

With the above notation, we may put

$$g_{\mu\nu}=\overset{+}{g}_{\mu\nu}\chi^{+}+\overset{-}{g}_{\mu\nu}\chi^{-}. \tag{3.8}$$

If we impose the conditions

$$[g_{\mu\nu}]=0 \tag{3.9}$$

then from the usual formulae

$$\begin{aligned}&\Gamma_{\mu\nu\sigma}=\tfrac{1}{2}(g_{\mu\nu,\sigma}+g_{\mu\sigma,\nu}-g_{\nu\sigma,\mu})\\&R_{\mu\nu}=\Gamma^{\alpha}_{\mu\alpha,\nu}-\Gamma^{\alpha}_{\mu\nu,\alpha}-\Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}+\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\\&T^{\mu\nu}=R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R\\&T^{\mu\nu}{}_{:\nu}=0\end{aligned} \tag{3.10}$$

we obtain, in view of (3.4) and (3.9),

$$\Gamma^{\mu}_{\nu\sigma}=\Gamma^{\mu+}_{\nu\sigma}\chi^{+}+\Gamma^{\mu-}_{\nu\sigma}\chi^{-}. \tag{3.11}$$

It follows that the Ricci tensor is given by

$$\begin{aligned}&R_{\mu\nu}=\overset{+}{R}_{\mu\nu}\chi^{+}+\overset{-}{R}_{\mu\nu}\chi^{-}+r_{\mu\nu}\\&r_{\mu\nu}=-[\Gamma^{\alpha}_{\mu\nu}]\chi^{+}_{,\alpha}+[\Gamma^{\alpha}_{\mu\alpha}]\chi^{+}_{,\nu}\end{aligned} \tag{3.12}$$

and that the material energy tensor is given by

$$\begin{aligned}&T^{\mu\nu}=T^{\mu\nu}_{+}\chi^{+}+T^{\mu\nu}_{-}\chi^{-}+t^{\mu\nu}\\&t^{\mu\nu}=r^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}r\\&r=g^{\mu\nu}r_{\mu\nu}.\end{aligned} \tag{3.13}$$

The conservation equations *at* Σ can now be stated in the usual form

$$t^{\mu\nu}{}_{:\nu}=0. \tag{3.14}$$

Some comments on (3.14) are in order. Firstly, although some of the Christoffel symbols are discontinuous and, therefore, undefined at Σ, (3.14) still makes sense because only products of the Christoffel symbols with δ_{Σ} appear in (3.14) and these are well-defined.

Secondly, (3.14) and (3.9) may be regarded as a set of equations which fix Cauchy data on Σ, for the Einstein equations in, say, V^{+}, provided the gravitational field is known in V^{-}: if $t_{\mu\nu}\neq 0$, (3.14) is a set of eight equations of the form

$$a^{\mu}\delta'+b^{\mu}\delta=0 \qquad \text{i.e. } a^{\mu}=0 \quad b^{\mu}=0. \tag{3.15}$$

Together with (3.9), these conditions are not sufficient to fix Cauchy data, unless it is possible to transform to a system of coordinates in which fewer of the $g_{\mu\nu,\sigma}$ have jump discontinuities across Σ. In gaussian coordinates, for example, only six additional conditions are required. However, in general, if the $g_{\mu\nu,\sigma}$ are discontinuous across Σ, gaussian coordinates can only be obtained by means of a transformation that is C^2 in V^+ and V^-, but only C^1 on Σ (Synge 1960).

This raises the question: are the equations (3.14) covariant under a transformation that is C^2 in V^+ and V^-, but only C^1 on Σ? Clearly, under admissible transformations $r_{\mu\nu}$ is a tensor, because $[\Gamma^{\mu}_{\nu\sigma}]$, being the difference of two affine connections, is a tensor, and χ^+ is a scalar density on account of the chain rule. Therefore, (3.14) is covariant under admissible transformations.

To deal with a transformation that is only C^1 on Σ, we make an intermediate transformation to the coordinates (x^0, z, x^2, x^3) where $z = f(x)$. The expression for $r_{\mu\nu}$ takes the form

$$r_{\mu\nu} = [\Gamma^{\alpha}_{\mu\alpha}]\chi^{+}_{,\nu} - [\Gamma^{1}_{\mu\nu}]\delta(z). \tag{3.16}$$

The first term on the right-hand side transforms correctly, since it is obtained from the product of the derivatives of two scalar densities. The second term on the right-hand side transforms correctly provided

$$[\partial^2 z/\partial \bar{x}^{\mu} \partial \bar{x}^{\nu}] = 0 \tag{3.17}$$

where $\bar{x}$ denotes the transformed coordinates. More generally, (3.14) will be covariant if the left-hand side of (3.17) adds, to the material energy tensor, a term with vanishing formal divergence.

This generalisation is not very useful, however, since it is not always possible to obtain gaussian coordinates using a transformation which satisfies these conditions. To ensure that the Cauchy problem remains well-posed, it is therefore necessary to restrict the class of coordinates to those that can be obtained from gaussian coordinates by admissible transformations (natural coordinates), or by transformations that are only C^1 on Σ but satisfy (3.17). With these restrictions on the coordinates, (3.14) becomes an overdetermined system, indicating that certain consistency conditions have to be satisfied for the existence of hypersurfaces of discontinuity.

While the present approach has the disadvantage that it is not coordinate-free, it has the advantage that it can be extended to situations where Israel's (1966) geometric approach is not applicable. To see this, we consider the case where Σ does not inherit a unique geometry from V^+ and V^-. In this situation, in natural coordinates, the conditions (3.9) no longer apply. The Christoffel symbols now take on the form

$$\Gamma^{\mu}_{\nu\sigma} = \Gamma^{\mu+}_{\nu\sigma}\chi^{+} + \Gamma^{\mu-}_{\nu\sigma}\chi^{-} + \gamma^{\mu}_{\nu\sigma}\delta_{\Sigma}. \tag{3.18}$$

Although the commutative law fails, in general, the Ricci tensor continues to be well-defined because $\delta_{\Sigma} \cdot \chi^{+} = \chi^{+} \cdot \delta_{\Sigma}$. Using formulae of the type

$$\chi^{+} \cdot \delta'_{\Sigma} = \tfrac{1}{2}\delta'_{\Sigma} - \delta^2_{\Sigma} \qquad \chi^{+} \cdot \delta^2_{\Sigma} = \tfrac{1}{2}\delta^2_{\Sigma} \tag{3.19}$$

in addition to (3.4), the distribution part of the material energy tensor takes on the general form

$$t^{\mu\nu} = a^{\mu\nu}\delta_{\Sigma} + b^{\mu\nu}\delta'_{\Sigma} + c^{\mu\nu}\delta^2_{\Sigma}. \tag{3.20}$$

Since $\delta_\Sigma \cdot \delta'_\Sigma \neq \delta'_\Sigma \cdot \delta_\Sigma$, the conservation equations (3.14) can now be interpreted in three inequivalent ways:

$$t^{\mu\nu}{}_{,\nu} + \Gamma^{\mu}_{\alpha\nu} t^{\alpha\nu} + \Gamma^{\nu}_{\alpha\nu} t^{\mu\alpha} = 0 \tag{3.21}$$

$$t^{\mu\nu}{}_{,\nu} + t^{\alpha\nu} \Gamma^{\mu}_{\alpha\nu} + \Gamma^{\nu}_{\alpha\nu} t^{\mu\alpha} = 0 \tag{3.22}$$

$$t^{\mu\nu}{}_{,\nu} + \Gamma^{\mu}_{\alpha\nu} \odot t^{\alpha\nu} + \Gamma^{\nu}_{\alpha\nu} \odot t^{\mu\alpha} = 0 \tag{3.23}$$

where $(\delta^2_\Sigma)' = 2\delta_\Sigma \odot \delta'_\Sigma$ in (3.23). If one of (3.21) or (3.22) is satisfied, so is (3.23), but all three equations are equivalent if and only if

$$\gamma^{\mu}_{\alpha\nu} b^{\alpha\nu} = \gamma^{\nu}_{\alpha\nu} b^{\mu\alpha} = 0 \tag{3.24}$$

where $\gamma^{\mu}_{\nu\sigma}$ and $b^{\mu\nu}$ are given by (3.18) and (3.20). The conditions (3.24) are identically satisfied, for instance, when Σ is a null surface and $t^{\mu\nu} = 0$, i.e. in the case of a strong gravitational shock. In view of (2.3), the equations (3.23) and (3.24) together correspond to a set of 32 equations, and form an overdetermined system.

The above considerations are of a formal nature, since we have not shown the existence of discontinuous solutions of the Einstein equations and the conservation equations. The problem regarding the existence of discontinuous solutions will be considered separately, and in this paper we only establish, quite rigorously, the existence of C^0 solutions. Before concluding this section, we remark that the present approach can also be used to study the case where the hypersurface of discontinuity is not smooth and has, for example, a 'corner' at some point. However, it seems best to postpone a discussion of this case till more results are available for the Cauchy problem in non-smooth domains.

4. The spherically symmetric surface layer

4.1. Preliminaries

The above techniques are primarily useful when, as in the case of shock waves, the $g_{\mu\nu}$ are known, *a priori*, only on one side of the hypersurface of discontinuity. In the case of the spherically symmetric surface layer the $g_{\mu\nu}$ on both sides are fixed by the requirements of symmetry and continuity. The general techniques, therefore, are not required for this special case. However, this simple case provides a C^0 solution of the Einstein equations and illustrates the manner in which the present techniques can be put to use.

The problem, conventionally, is to determine the motion of the spherically symmetric surface layer of matter at the junction between the Schwarzschild and Minkowski metrics. It is assumed that the $g_{\mu\nu}$ are continuous across the hypersurface of discontinuity, and that some of their first derivatives have discontinuities. We will adopt the radiative coordinates used by Papapetrou and Hamoui (1968): $x^0 = r$, $x^1 = u$, $x^2 = \theta$, $x^3 = \varphi$. We assume that the hypersurface of discontinuity, Σ, is spacelike, so that its equation can always be written (locally) in the form

$$r = f(u). \tag{4.1}$$

We also suppose that $r > 2m$, so that the external metric has no singularity.

The external and internal metrics, respectively, are given by

$$\begin{aligned} ds_+^2 &= (1-2m/r)\,du^2+2du\,dr-r^2(d\theta^2+\sin^2\theta\,d\varphi^2) \qquad r>f(u) \\ dS_-^2 &= dU^2+2dU\,dR-R^2(d\theta^2+\sin^2\theta\,d\varphi^2) \qquad R<F(U) \end{aligned} \tag{4.2}$$

where $R=F(U)$ is the equation of the hypersurface in the interior coordinates (R, U, θ, φ). Carrying out the change of variables $r\to z=r-f(u)$, $R\to Z=R-F(U)$, the external and internal metrics may be written as

$$\begin{aligned} ds_+^2 &= \left(1-\frac{2m}{z+f}+2f'\right)du^2+2du\,dz-(z+f)^2(d\theta^2+\sin^2\theta\,d\varphi^2) \qquad z>0 \\ dS_-^2 &= (1+2F')\,dU^2+2dU\,dZ-(Z+F)^2(d\theta^2+\sin^2\theta\,d\varphi^2) \qquad Z<0. \end{aligned} \tag{4.3}$$

We suppose that the coordinates (U, Z, θ, φ) can be obtained from the coordinates (u, z, θ, φ) by the change of variables

$$\begin{aligned} U &= \alpha(u) \\ Z &= \gamma(u)z \qquad \gamma>0 \end{aligned} \tag{4.4}$$

and that the junction conditions

$$[g_{\mu\nu}]=0 \tag{4.5}$$

are satisfied. (4.4) and (4.5) together yield

$$\begin{aligned} &F(U)=f(u) \\ &\alpha'\gamma=1 \\ &\alpha'(\alpha'+2f')=1-2m/f+2f'. \end{aligned} \tag{4.6}$$

These equations were obtained by Papapetrou and Hamoui (1968).

In the present calculation we require the same coordinates throughout, and in the coordinates (u, z, θ, φ) the external and internal metrics are given (using the junction conditions) by

$$\begin{aligned} ds_+^2 &= \left(1-\frac{2m}{z+f}+2f'\right)du^2+2du\,dz-(z+f)^2(d\theta^2+\sin^2\theta\,d\varphi^2) \qquad z>0 \\ ds_-^2 &= \left(1-\frac{2m}{f}+2f'+2\alpha'\gamma'z\right)du^2+2du\,dz-(\gamma z+f)^2(d\theta^2+\sin^2\theta\,d\varphi^2) \qquad z<0. \end{aligned} \tag{4.7}$$

To simplify the derivation of the equations of motion, we put

$$\begin{aligned} a^+ &= a^+(u,z)=1-2m/(z+f)+2f' \\ a^- &= a^-(u,z)=1-2m/f+2f'+2\alpha'\gamma'z \\ b^+ &= b^+(u,z)=-(z+f)^2 \\ b^- &= b^-(u,z)=-(\gamma z+f)^2 \\ a &= a^+\chi^++a^-\chi^- \\ b &= b^+\chi^++b^-\chi^-. \end{aligned} \tag{4.8}$$

Denoting $a_{,1}$ by a_1, etc, it follows that

$$\begin{aligned} &[a]=0 && [b]=0 \\ &[a_0]=0 && [b_0]=0 \qquad (4.9) \\ &[a_1]=2(m/f^2-\alpha'\gamma') && [b_1]=2f(\gamma-1) \\ &a|=1-2m/f+2f' && b|=-f^2 \\ &a_0|=2mf'/f^2+2f'' && b_0|=-2ff' \qquad (4.10) \\ &a_1|=m/f^2+\alpha'\gamma' && b_1|=-f(\gamma+1). \end{aligned}$$

The junction conditions (4.5) have been used in writing down the above equations.

4.2. *The equations of motion*

Because of the new notation and techniques, the various steps in the derivation of the equations of motion are given below in detail. In (4.11)–(4.13) the superscripts $^+$ and $^-$ have been omitted with the understanding that either superscript may be used consistently in any equation. In the new notation, the non-vanishing components of the metric tensor are given by

$$\begin{aligned} &g_{00}=a && g_{22}=b \\ &g_{01}=1 && g_{33}=b\sin^2\theta \qquad (4.11) \\ &g^{01}=1 && g^{22}=b^{-1} \\ &g^{11}=a && g^{33}=b^{-1}\sin^{-2}\theta. \qquad (4.12) \end{aligned}$$

Similarly, the non-vanishing Christoffel symbols are given by

$$\begin{aligned} &\Gamma^0_{00}=-\tfrac{1}{2}a_1 && \Gamma^2_{02}=\tfrac{1}{2}b^{-1}b_0 \\ &\Gamma^0_{22}=-\tfrac{1}{2}b_1 && \Gamma^2_{12}=\tfrac{1}{2}b^{-1}b_1 \\ &\Gamma^0_{33}=\Gamma^0_{22}\sin^2\theta && \Gamma^2_{33}=-\sin\theta\cos\theta \\ &\Gamma^1_{00}=\tfrac{1}{2}(aa_1+a_0) && \Gamma^3_{03}=\tfrac{1}{2}b^{-1}b_0 \qquad (4.13) \\ &\Gamma^1_{01}=\tfrac{1}{2}a_1 && \Gamma^3_{13}=\tfrac{1}{2}b^{-1}b_1 \\ &\Gamma^1_{22}=\tfrac{1}{2}(ab_1-b_0) && \Gamma^3_{23}=\cot\theta \\ &\Gamma^1_{33}=\Gamma^1_{22}\sin^2\theta. && \end{aligned}$$

The distribution part of the Ricci tensor is given, according to (3.12), by

$$\begin{aligned} &r_{00}=[\Gamma^1_{00}]\delta=-\tfrac{1}{2}a\,|[a_1]\delta \\ &r_{01}=[\Gamma^0_{00}+2\Gamma^2_{02}]\delta=-\tfrac{1}{2}[a_1]\delta=r_{10} \\ &r_{11}=2[\Gamma^2_{12}]\delta=b^{-1}|[b_1]\delta \qquad (4.14) \\ &r_{22}=-[\Gamma^1_{22}]\delta=-\tfrac{1}{2}a\,|[b_1]\delta \\ &r_{33}=-[\Gamma^1_{33}]\delta=r_{22}\sin^2\theta. \end{aligned}$$

The same tensor, with raised suffixes, is given by

$$\begin{aligned} r^{00} &= b^{-1}|[b_1]\delta \\ r^{01} &= -\tfrac{1}{2}[a_1+2ab^{-1}b_1]\delta = r^{10} \\ r^{11} &= \tfrac{1}{2}[aa_1+2a^2b^{-1}b_1]\delta \\ r^{22} &= -\tfrac{1}{2}ab^{-2}|[b_1]\delta \\ r^{33} &= r^{22}/\sin^2\theta \end{aligned} \tag{4.15}$$

and its trace by

$$r = -2ab^{-1}|[b_1]\delta - 2[a_1]\delta. \tag{4.16}$$

The components of the material energy tensor are given by

$$\begin{aligned} t^{00} &= b^{-1}|[b_1]\delta \\ t^{01} &= 0 = t^{11} \\ t^{22} &= \tfrac{1}{2}ab^{-2}|[b_1]\delta + \tfrac{1}{2}b^{-1}|[a_1]\delta \\ t^{33} &= t^{22}/\sin^2\theta. \end{aligned} \tag{4.17}$$

The last two of the equations (3.14) are identically satisfied, and the other two are

$$\begin{aligned} t^{0\nu}{}_{:\nu} &= t^{00}{}_{,0} + (\Gamma^0_{00}+\Gamma^\nu_{0\nu})t^{00} + 2\Gamma^0_{22}t^{22} = 0 \\ t^{1\nu}{}_{:\nu} &= \Gamma^1_{00}t^{00} + 2\Gamma^1_{22}t^{22} = 0. \end{aligned} \tag{4.18}$$

Explicitly, we have the equations

$$\begin{aligned} &(b^{-1}|[b_1])_{,0} - \tfrac{1}{2}a_1b^{-1}|[b_1] - \tfrac{1}{2}ab^{-2}b_1|[b_1] + b^{-2}b_0|[b_1] - \tfrac{1}{2}b^{-1}b_1|[a_1] = 0 \\ &(aa_1+a_0)|[b_1] + (ab_1-b_0)|[ab^{-1}b_1+a_1] = 0 \end{aligned} \tag{4.19}$$

along with (4.6) which can be restated as

$$\begin{aligned} \alpha'\gamma &= 1 \\ \alpha'(\alpha'+2f') &= a|. \end{aligned} \tag{4.20}$$

We show below that the first two equations, (4.19), reduce to identities by virtue of (4.20).

Thus, the first of (4.19) can be written in the form

$$a|(\gamma^2-1) - 2(\gamma-1)f' = 2m/f \tag{4.21}$$

where we have used the values of the bars and brackets, given in (4.9) and (4.10). From (4.20)

$$a|\gamma^2 - 2f'\gamma - 1 = 0. \tag{4.22}$$

Substituting in (4.21) we have an identity by virtue of the value of $a|$ given in (4.10). Similarly, the second of (4.19) can be shown to be an identity by repeated use of (4.22).

Therefore, for the case under consideration the equations (3.14) are identical with the equations (3.9), and it has been rigorously established that the spherically symmetric surface layer at the Schwarzschild–Minkowski junction provides a C^0 solution of the Einstein equations and the conservation equations.

We observe that the equations of motion are underdetermined, and an extra condition, like an equation of state, is required to specify the motion of the hypersurface completely. There is no need to go into this problem here, as our conclusions are in agreement with the conclusions reached by earlier approaches, and the motion of the hypersurface has been studied, in detail, by Israel (1966), Papapetrou and Hamoui (1968, 1979) and Evans (1977).

5. Relativistic characterisation of matter

In the above sections we analysed the problem of junction conditions from a mathematical standpoint, and concluded that the equations of relativity remain meaningful even if the smoothness conditions on the $g_{\mu\nu}$ are relaxed. In this section we propose to adopt a more physical point of view: what degree of smoothness for the $g_{\mu\nu}$ is compatible with observed matter distributions? No serious attempt has been made to answer this question, since discontinuities or singularities in the matter tensor are usually permitted, as a matter of convenience, to simplify the study of hypersurfaces across which there are large changes in the $g_{\mu\nu,\sigma}$; and it is often stated (e.g. Hawking and Ellis 1973) that one may reasonably assume the $g_{\mu\nu}$ to be of class C^2 globally.

It is possible to give, at least, a partial answer to the question posed above by appealing to the fact that matter exists in discrete clusters at the microscopic level. This procedure raises a fundamental issue. Since the nexus between relativity and elementary particle theory is so hazy as to be almost imperceptible, at present, can it be seriously asserted that relativity is at all relevant to a description of the distribution of matter corresponding to an elementary particle? In fact, is there any justification for extrapolating relativity to the attoscopic level?

According to the prevalent view, there is no such justification, and relativity may be used at the microscopic level only after it has been successfully integrated with quantum field theory. Objectively, the prevalent view appears to be fallacious on two counts. Firstly, the classical theory of relativity is formulated without regard to scale, and there is no definite evidence for its failure at the microscopic level. The prevalent view, therefore, condemns relativity, at the microscopic level, without a trial.

Secondly, unlike the electromagnetic field, no theoretical inconsistency would arise if the gravitational field were left unquantised. In fact, since the notion of a space–time manifold is a primitive notion underlying the formalism of conventional quantum theory, if the gravitational field is quantised, quantum theory would have to be modified alongside. Therefore, a decision regarding the quantisation of the gravitational field would seem to be best based on an interpretation of quantum mechanics.

We propose to use Raju's (1981) interpretation of quantum mechanics as a semiclassical theory of extended particles. According to this interpretation, quantum mechanical particles have an extended shell-like structure, and relativity, assumed to be valid at the microscopic level, can be used to examine this structure. This interpretation does postulate the occurrence of certain fluctuations in the metric tensor, against a smoothed-out background. While these fluctuations are not specifically quantum mechanical in origin, they do lead to analogues of quantum mechanical behaviour. With this interpretation of quantum mechanics, the subsequent results could be regarded as referring to the averaged-out background metric.

In a relativistic characterisation of the distribution of matter corresponding to an elementary particle, with non-zero rest mass, the simplest possibility which arises is that of a point mass. This possibility has been rejected by Dirac (1962) on the following grounds: a point mass would behave like a black hole, real particles do not behave like black holes, hence real particles are not point masses. However, given the empirically determined masses and charges of real elementary particles, the Reissner–Nordström solution assures us that the above argument is not applicable to real, charged particles. Given, further, that real particles are often charged, Dirac's (1962) argument loses much of its significance. Nevertheless, even in the Reissner–Nordström metric, there is a singularity at the origin, so, even if we were to accept the point of view that real particles are pointlike, the $g_{\mu\nu}$ would not be continuous everywhere.

The other possibility is that of an extended mass distribution. The stability of such a mass distribution is a difficult problem, especially in the case of charged particles, because of the inordinate imbalance between gravitational and electromagnetic forces. Given that electrons, for instance, are reasonably stable, if we subscribe to the view that real particles are extended, we seem to be faced with the alternatives of introducing new phenomenology or abandoning the existing theory of relativity at the microscopic level. However, in view of the earlier results on stationary surface layers (e.g. Papapetrou and Hamoui 1968) there is a third possibility; namely, the $g_{\mu\nu}$ may not be of class C^1 as is usually supposed. If we accept the theory of relativity, and Occam's razor, we are forced to accept the third possibility.

The conclusions reached above may be summarised by saying that the theory of relativity, in its present form, is consistent with the existence of particles only if the $g_{\mu\nu}$ are (globally) *at most* of class C^0. This conclusion calls for some rethinking on the postulates of the Hawking–Penrose singularity theory. Much of the work in singularity theory uses hypotheses which imply that the $g_{\mu\nu}$ are *at least* of class C^1 (Hawking and Ellis 1973). Therefore, according to the above arguments, these results are not consistent with the existence of matter in the form of particles and, hence, are not applicable to the real universe.

6. Conclusions

The equations of relativity remain meaningful even if the $g_{\mu\nu}$ have a simple discontinuity across a smooth hypersurface, in natural coordinates. If the theory of relativity is valid at the microscopic level, it is likely that the $g_{\mu\nu}$ are not globally continuously differentiable.

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